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COMPLETE CONVERGENCE FOR THE MAXIMAL PARTIAL SUMS WITHOUT MAXIMAL INEQUALITIES

Fakhreddine Boukhari

Laboratoire de Statistique et Modélisations Aléatoires, Faculty of Sciences, Abou Bekr Belkaid University, BP 119, Tlemcen 13000, Algeria.
E-Mail f_boukhari@yahoo.fr

Nguyen Chi Dzung

Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Hanoi 10307, Vietnam.
E-Mail nczung@gmail.com

Lê Văn Thành*

Department of Mathematics, Vinh University, 182 Le Duan, Vinh, Nghe An, Vietnam.
E-Mail levt@vinhuni.edu.vn

Abstract. This work provides the necessary and sufficient conditions for complete convergence for the maximal partial sums of dependent random variables. The results are proved without using maximal inequalities. The main theorems can be applied to sequences of (i) m-pairwise negatively dependent random variables and (ii) m-extended negatively dependent random variables. While the result for case (i) unifies and improves many existing ones, the result for case (ii) complements the main theorem of Chen et al. [J. Appl. Probab., 2010]. Affirmative answers to open questions raised by Chen et al. [J. Math. Anal. Appl., 2014], and Wu and Rosalsky [Glas. Mat. Ser. III, 2015] are also given. Two examples illustrating the sharpness of the main result are presented.

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1. Introduction and the main result. This work is an improvement of the arXiv preprint [10]. Let $\{X, X_n, n \geq 1\}$ be a sequence of pairwise independent and identically distributed (p.i.i.d.) random variables. Etemadi [11] is the first author who proved the Kolmogorov strong law of large numbers (SLLN)

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n}(X_i - \mathbb{E}X_i)}{n} = 0 \text{ almost surely (a.s.)}$$

*Corresponding author.

under the optimal moment condition $\mathbb{E}|X| < \infty$ without using the maximal inequalities. The Etemadi result was further extended to random fields by Fazekas and Tómács [12] in which the authors also obtained the rate of convergence. The problem of proving the Marcinkiewicz–Zygmund SLLN for p.i.i.d. random variables under an optimal moment condition is more challenging, and Etemadi's method in [11] does not seem to work if the normalizing constants are of the form $n^{1/p}$ with $p > 1$. Let $1 < p < 2$. Martĭkainen [17] proved that if $\mathbb{E}(|X|^p \log^\beta |X|) < \infty$ for some $\beta > \max\{0, 4p-6\}$, then the Marcinkiewicz–Zygmund SLLN holds, i.e.,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n (X_i - \mathbb{E}X_i)}{n^{1/p}} = 0 \ \text{ a.s.} \tag{1.1}$$

Here and hereafter, for $x \geq 0$ and $\beta \in \mathbb{R}$, we denote the natural logarithm of $\max\{x, \mathrm{e}\}$ by $\log x$, and write $\log^\beta x = (\log x)^\beta$. As far as we know, Rio [20] is the first author who proved the Marcinkiewicz–Zygmund SLLN (1.1) under the optimal moment condition $\mathbb{E}|X|^p < \infty$. Anh et al. [1] recently proved the Marcinkiewicz–Zygmund-type SLLN with the norming constants of the form $n^{1/p}\tilde{L}(n^{1/p})$, $n \geq 1$, where $\tilde{L}(\cdot)$ is the de Bruijn conjugate of a slowly varying function $L(\cdot)$. However, the proof in [1] is based on a maximal inequality for negatively associated random variables which is no longer available even for pairwise independent random variables.

In this paper, we use Rio's method [20] and the theory of regularly varying functions to derive rates of convergence in the SLLN under optimal moment conditions. Although Rio's result was extended by Thành [24], it only considered sums for p.i.i.d. random variables. The motivation of the present paper is that many other dependence structures do not enjoy a Kolmogorov-type maximal inequality such as pairwise negative dependence, and extended negative dependence, among others (see, e.g., [4, 8, 22, 29] and the references therein). In contrast to [24], we explore the scenario where the involved family of random variables is not necessarily stochastically dominated and establish a Baum–Katz-type theorem under a uniformly bounded moment condition. We also provide a necessary condition for the convergence of the Baum–Katz series.

The main result of this paper is the following theorem. To our best knowledge, Theorem 1.1 and Corollary 1.2 are new even when the underlying sequence is comprised of independent random variables.

THEOREM 1.1. *Let $1 \leq p < 2$, and $\{X_n, n \geq 1\}$ be a sequence of random variables. Assume that there exists a universal constant C such that for all nondecreasing functions f_i, $i \geq 1$,*

$$\operatorname{Var}\left(\sum_{i=k+1}^{k+\ell} f_i(X_i)\right) \leq C \sum_{i=k+1}^{k+\ell} \operatorname{Var}(f_i(X_i)),\ k \geq 0, \ell \geq 1, \tag{1.2}$$

provided the variances exist. Let $L(\cdot)$ be a slowly varying function defined on $[0,\infty)$ and let $\tilde{L}(\cdot)$ be the de Bruijn conjugate of $L(\cdot)$. When $p = 1$, we assume further that $L(x) \geq 1$ and is increasing on $[0,\infty)$. If

$$\sup_{n\geq 1} \mathbb{E}\left(|X_n|^p L^p(|X_n|) \log(|X_n|) \log^2(\log|X_n|)\right) < \infty, \tag{1.3}$$

then for all $\alpha \geq 1/p$, *we have*

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(X_i-\mathbb{E}X_i)\right| > \varepsilon n^{\alpha}\tilde{L}(n^{\alpha})\right) < \infty \text{ for all } \varepsilon > 0. \tag{1.4}$$

Conversely, if

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(X_i-c_i)\right| > \varepsilon n^{\alpha}\tilde{L}(n^{\alpha})\right) < \infty \text{ for all } \varepsilon > 0, \tag{1.5}$$

where $\{c_i, i \geq 1\}$ *is a sequence of real numbers, then*

$$\sum_{n\geq 1} n^{\alpha p-2}\sum_{i=1}^{n}\mathbb{P}\big(|X_i-c_i| > n^{\alpha}\tilde{L}(n^{\alpha})\big) < \infty. \tag{1.6}$$

REMARK 1. (i) Many dependence structures enjoy (1.2) such as negative association, pairwise independence, pairwise negative dependence, extended negative dependence, various mixing sequences, etc. The SLLN for sequences and fields of random variables satisfying these dependence structures was studied by many authors. We refer to [2, 9, 12, 14, 15, 16, 19] and the references therein.

(ii) Theorem 1.1 can fail if Condition (1.2) is not satisfied (see Example 2.1).

(iii) Condition (1.3) is very sharp. Even when the involved random variables are independent, Example 3.1 in Section 3 shows that Theorem 1.1 may fail if (1.3) is weakened to

$$\sup_{n\geq 1}\mathbb{E}\left(|X_n|^p L^p(|X_n|)\log(|X_n|)\log(\log|X_n|)\right) < \infty.$$

Considering a special interesting case $\alpha = 1/p$ and $L(x) \equiv 1$, we obtain the following corollary.

COROLLARY 1.2. *Let* $1 \leq p < 2$, *and* $\{X_n, n \geq 1\}$ *be a sequence of random variables satisfying Condition* (1.2). *If*

$$\sup_{n\geq 1}\mathbb{E}\left(|X_n|^p\log(|X_n|)\log^2(\log|X_n|)\right) < \infty, \tag{1.7}$$

then

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{P}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(X_i-\mathbb{E}X_i)\right| > \varepsilon n^{1/p}\right) < \infty \text{ for all } \varepsilon > 0. \tag{1.8}$$

REMARK 2. (i) Since the sequence $\{\max_{1\leq j\leq n}|\sum_{i=1}^{j}(X_i-\mathbb{E}X_i)|, n \geq 1\}$ is nondecreasing, it follows from (1.8) that the SLLN (1.1) holds.

(ii) For the SLLN under the uniformly bounded moment condition, Baxter et al. [3] proved (1.1) with assumptions that the sequence $\{X_n, n \geq 1\}$ is independent and $\sup_{n\geq 1} \mathbb{E}|X_n|^r < \infty$ for some $r > p$. This condition is much stronger than (1.7). Baxter et al. [3] studied the SLLN for weighted sums and their method does not give the rate of convergence as in Corollary 1.2.

(iii) For sequence of p.i.i.d. random variables $\{X, X_n, n \geq 1\}$, Chen et al. [7] obtained (1.8) under the condition that $\mathbb{E}(|X|^p \log^r |X|) < \infty$ for some $1 < p < r < 2$. In Corollary 1.2, the moment Condition (1.7) is weaker than that of Chen et al. [7].

The rest of the paper is arranged as follows. Section 2 presents a complete convergence result for sequences of dependent random variables with regularly varying normalizing constants under a stochastic domination condition. The proof of Theorem 1.1 is given in Section 3. Finally, Section 4 contains corollaries and remarks comparing our results and the ones in the literature. As for the notation, we shall write $u_n = o(v_n)$ (resp., $u_n \asymp v_n$) to indicate that $u_n/v_n \to 0$ as n tends to infinity (resp., $c_1 u_n \leq v_n \leq c_2 u_n$ for large values of n and some positive constants $c_1,\ c_2$).

2. Complete convergence for the maximal partial sums of dependent random variables with regularly varying normalizing constants under a stochastic domination condition. In this section, we will use Rio's method [20] to obtain complete convergence for sums of dependent random variables with regularly varying constants under a stochastic domination condition.

A family of random variables $\{X_i, i \in I\}$ is said to be stochastically dominated by a random variable X if

$$\sup_{i\in I} \mathbb{P}(|X_i| > t) \leq \mathbb{P}(|X| > t), \quad \text{for all } t \geq 0. \tag{2.1}$$

We note that many authors use an apparently weaker definition of $\{X_i, i \in I\}$ being stochastically dominated by a random variable Y, namely that

$$\sup_{i\in I} \mathbb{P}(|X_i| > t) \leq C\mathbb{P}(|Y| > t), \text{ for all } t \geq 0 \tag{2.2}$$

for some constant $C \in (0, \infty)$. However, it is shown by Rosalsky and Thành [21] that (2.1) and (2.2) are indeed equivalent.

Let $\rho \in \mathbb{R}$. A real-valued function $R(\cdot)$ is said to be *regularly varying* (at infinity) with index of regular variation ρ if it is a positive and measurable function on $[A, \infty)$ for some $A \geq 0$, and for each $\lambda > 0$,

$$\lim_{x\to\infty} \frac{R(\lambda x)}{R(x)} = \lambda^\rho.$$

A regularly varying function with the index of regular variation $\rho = 0$ is called *slowly varying* (at infinity). If $L(\cdot)$ is a slowly varying function, then by Theorem 1.5.13 in Bingham et al. [5], there exists a slowly varying function $\tilde{L}(\cdot)$, unique up to asymptotic equivalence, satisfying

$$\lim_{x\to\infty} L(x)\tilde{L}(xL(x)) = 1 \text{ and } \lim_{x\to\infty} \tilde{L}(x)L\left(x\tilde{L}(x)\right) = 1. \tag{2.3}$$

The function $\tilde{L}$ is called the de Bruijn conjugate of L, and $\left(L, \tilde{L}\right)$ is called a (slowly varying) conjugate pair (see, e.g., p. 29 in Bingham et al. [5]). If $L(x) = \log^\gamma x$ or $L(x) = \log^\gamma(\log x)$ for some $\gamma \in \mathbb{R}$, then $\tilde{L}(x) = 1/L(x)$. Especially, if $L(x) \equiv 1$, then $\tilde{L}(x) \equiv 1$.

Here and thereafter, for a slowly varying function $L(\cdot)$, we denote the de Bruijn conjugate of $L(\cdot)$ by $\tilde{L}(\cdot)$. Throughout, we will assume, without loss of generality, that $L(x)$ and $\tilde{L}(x)$ are both continuous on $[0,\infty)$, differentiable on $[A,\infty)$ for some $A \geq 0$, and $x^\gamma L(x)$ and $x^\gamma \tilde{L}(x)$ are both strictly increasing on $[0,\infty)$ for all $\gamma > 0$ (see Thành [26, p. 578]). We also assume that (see Lemma 2.2 in Anh et al. [1])

$$\lim_{x\to\infty} \frac{xL'(x)}{L(x)} = 0 \text{ and } \lim_{x\to\infty} \frac{x\tilde{L}'(x)}{\tilde{L}(x)} = 0. \tag{2.4}$$

The following theorem establishes complete convergence for the maximal partial sums of dependent random variables without using the Kolmogorov-type maximal inequalities. For the special case where $L(x) = \log^\alpha x$ with $\alpha \geq 0$, Miao et al. [18] proved Theorem 2.1 for sequences of negatively associated random variables, which do enjoy the Kolmogorov maximal inequality. The main contribution of our result is that it can be applied to dependence structures where the Kolmogorov-type maximal inequalities may not hold.

Theorem 2.1. *Let $1 \leq p < 2$ and let $\{X_n, n \geq 1\}$ be a sequence of random variables satisfying Condition* (1.2). *Let $L(\cdot)$ be as in Theorem 1.1. If $\{X_n,\, n \geq 1\}$ is stochastically dominated by a random variable X, and*

$$\mathbb{E}\left(|X|^p L^p(|X|)\right) < \infty, \tag{2.5}$$

then for all $\alpha \geq 1/p$, we have

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \mathbb{P}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(X_i - \mathbb{E}X_i)\right| > \varepsilon n^\alpha \tilde{L}(n^\alpha)\right) < \infty \text{ for all } \varepsilon > 0. \tag{2.6}$$

We only sketch the proof of Theorem 2.1 and refer the reader to the proof of Theorem 1 in Thành [24] for details. The main difference here is that we have to consider the nonnegative random variables so that after applying certain truncation techniques (see (2.8) and (2.9) below), the new random variables still satisfy Condition (1.2).

Sketch proof of Theorem 2.1. Since $\{X_n^+, n \geq 1\}$ and $\{X_n^-, n \geq 1\}$ satisfy the assumptions of the theorem and $X_n = X_n^+ - X_n^-, n \geq 1$, without loss of generality we can assume that $X_n \geq 0$ for all $n \geq 1$. For $n \geq 1$, set

$$b_n = n^\alpha \tilde{L}\left(n^\alpha\right), \tag{2.7}$$

$$X_{i,n} = X_i \mathbf{1}(X_i \leq b_n) + b_n \mathbf{1}(X_i > b_n),\ 1 \leq i \leq n, \tag{2.8}$$

and

$$Y_{i,m} = \left(X_{i,2^m} - X_{i,2^{m-1}}\right) - \mathbb{E}\left(X_{i,2^m} - X_{i,2^{m-1}}\right),\ m \geq 1,\ i \geq 1. \tag{2.9}$$

Since $x\tilde{L}(x)$ is strictly increasing on $[0,\infty)$, $\{b_n, n \geq 0\}$ is strictly increasing sequence. It is easy to see that (2.6) is equivalent to

$$\sum_{n=1}^{\infty} 2^{n(\alpha p-1)} \mathbb{P}\left(\max_{1\leq j<2^n} \left| \sum_{i=1}^{j} (X_i - \mathbb{E}X_i) \right| > \varepsilon b_{2^n} \right) < \infty \text{ for all } \varepsilon > 0. \tag{2.10}$$

It follows from the stochastic domination condition and definition of b_n that

$$0 \leq \mathbb{E}\left(X_{i,2^m} - X_{i,2^{m-1}}\right) \leq \mathbb{E}\left(|X|\mathbf{1}(|X| > b_{2^{m-1}})\right). \tag{2.11}$$

Using (2.11) and proceeding in a similar manner as in Thành [24, Equation (23)], the proof of (2.10) will be completed if we can show that

$$\sum_{n=1}^{\infty} 2^{n(\alpha p-1)} \mathbb{P}\left(\max_{1\leq j<2^n} \left| \sum_{i=1}^{j} (X_{i,2^n} - \mathbb{E}X_{i,2^n}) \right| \geq \varepsilon b_{2^{n-1}} \right) < \infty \text{ for all } \varepsilon > 0. \tag{2.12}$$

For $m \geq 0$, set $S_{0,m} = 0$ and

$$S_{j,m} = \sum_{i=1}^{j} (X_{i,2^m} - \mathbb{E}X_{i,2^m}),\ j \geq 1.$$

For $1 \leq j < 2^n$ and for $0 \leq m \leq n$, let $k_{j,m} = \lfloor j/2^m \rfloor$ be the greatest integer which is less than or equal to $j/2^m$, $j_m = k_{j,m}2^m$. Then (see Thành [24, Equation (28)])

$$\begin{aligned} \max_{1\leq j<2^n} |S_{j,n}| \leq & \sum_{m=1}^{n} \max_{0\leq k<2^{n-m}} \left| \sum_{i=k2^m+1}^{k2^m+2^{m-1}} \left(X_{i,2^{m-1}} - \mathbb{E}X_{i,2^{m-1}}\right) \right| \\ & + \sum_{m=1}^{n} \max_{0\leq k<2^{n-m}} \left| \sum_{i=k2^m+1}^{(k+1)2^m} Y_{i,m} \right| + \sum_{m=1}^{n} 2^{m+1} \mathbb{E}\left(|X|\mathbf{1}(|X| > b_{2^{m-1}})\right). \end{aligned} \tag{2.13}$$

Combining (1.2), (2.8) and (2.9), we have for all $m \geq 1$,

$$\mathbb{E}\left(\sum_{i=k+1}^{k+\ell} \left(X_{i,2^{m-1}} - \mathbb{E}X_{i,2^{m-1}}\right) \right)^2 \leq C \sum_{i=k+1}^{k+\ell} \mathbb{E}X_{i,2^{m-1}}^2,\ k \geq 0, \ell \geq 1, \tag{2.14}$$

and

$$\mathbb{E}\left(\sum_{i=k+1}^{k+\ell} Y_{i,m} \right)^2 \leq C \sum_{i=k+1}^{k+\ell} \mathbb{E}Y_{i,m}^2,\ k \geq 0, \ell \geq 1. \tag{2.15}$$

By using (2.13)–(2.15) and proceeding in a similar manner as in pages 1236-1238 in Thành [24], we obtain (2.12). □

Remark 3. When $0 < p < 1$, we can show that Theorems 1.1 and 2.1 hold irrespective of the dependence structure of the underlying sequence of random variables (see Theorems 3.1 and 3.2 of Boukhari [6]). However, for the case $1 \leq p < 2$, the following simple example shows that these theorems can fail if the involved random variables do not satisfy (1.2).

Example 2.1. Let $X_n \equiv X$, where X is a Bernoulli random variable with $\mathbb{P}(X = \pm 1) = 1/2$. It is easy to see that Condition (1.2) fails. Let $1 \leq p < 2$ and consider the case where $L(x) \equiv 1$ and $\alpha = 1/p \leq 1$. Since X is a bounded random variable, Conditions (1.3) and (2.5) are both satisfied. We have with probability 1 that

$$|X_1 + \cdots + X_n| = n|X| = n \geq n^{\alpha}$$

and therefore for all $0 < \varepsilon \leq 1$,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(X_i - \mathbb{E}X_i)\right| > \varepsilon n^{\alpha}\tilde{L}(n^{\alpha})\right) &= \sum_{n=1}^{\infty} n^{-1}\mathbb{P}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} X_i\right| > \varepsilon n^{\alpha}\right) \\ &\geq \sum_{n=1}^{\infty} n^{-1} = \infty.\end{aligned}$$

The next theorem shows that the moment condition in (2.5) in Theorem 2.1 is optimal.

Theorem 2.2. *Let $1 \leq p < 2$, $1/p \leq \alpha \leq 1$ and let $\{X,\ X_n,\ n \geq 1\}$ be a sequence of identically distributed random variables satisfying* (1.2). *Let $L(\cdot)$ be a slowly varying function defined on $[0, \infty)$. When $\alpha = 1$, we assume further that $L(x) \geq 1$ and is increasing on $[0, \infty)$. If for some constant c,*

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(X_i - c)\right| > \varepsilon n^{\alpha}\tilde{L}(n^{\alpha})\right) < \infty \text{ for all } \varepsilon > 0, \tag{2.16}$$

then $\mathbb{E}\left(|X|^p L^p(|X|)\right) < \infty$ and $\mathbb{E}X = c$.

Proof. Let $b_n = n^{\alpha}\tilde{L}(n^{\alpha}), n \geq 1$. Note again that we can assume that b_n is strictly increasing (see, e.g., Proposition B.1.9 in [13]). A direct application of the second portion of Theorem 1.1 with $c_i \equiv c$ yields

$$\sum_{n\geq 1} n^{\alpha p-1}\mathbb{P}\left(|X - c| > b_n\right) < \infty.$$

Employing Proposition 2.6 of [1], we obtain

$$\mathbb{E}\left(|X - c|^p L^p(|X - c|)\right) < \infty. \tag{2.17}$$

Since $L(\cdot)$ is slowly varying and c is a constant, (2.17) implies $\mathbb{E}\left(|X|^p L^p(|X|)\right) < \infty$. Applying Theorem 2.1, we obtain

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(X_i - \mathbb{E}X_i)\right| > \varepsilon b_n\right) < \infty \text{ for all } \varepsilon > 0. \tag{2.18}$$

Let $\varepsilon > 0$ be arbitrary. Since $\alpha p \geq 1$ and $0 < b_n \uparrow$, we have from (2.18) that

$$
\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(X_i-\mathbb{E}X_i)\right| > \varepsilon b_n\right)\\
&\geq \sum_{n=1}^{\infty} n^{-1}\mathbb{P}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(X_i-\mathbb{E}X_i)\right| > \varepsilon b_n\right)\\
&= \sum_{k=1}^{\infty}\sum_{n=2^{k-1}}^{2^k-1} n^{-1}\mathbb{P}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(X_i-\mathbb{E}X_i)\right| > \varepsilon b_n\right)\\
&\geq \frac{1}{2}\sum_{k=1}^{\infty}\mathbb{P}\left(\max_{1\leq j\leq 2^{k-1}}\left|\sum_{i=1}^{j}(X_i-\mathbb{E}X_i)\right| > \varepsilon b_{2^k}\right).
\end{aligned} \tag{2.19}
$$

By applying the Borel–Cantelli lemma, (2.19) implies

$$
\lim_{k\to\infty}\frac{\max_{1\leq j\leq 2^{k-1}}\left|\sum_{i=1}^{j}(X_i-\mathbb{E}X_i)\right|}{b_{2^k}} = 0 \text{ a.s.} \tag{2.20}
$$

It is clear that $b_{2n}/b_n \asymp 1$. Therefore, we infer from (2.20) and the identical distribution assumption that

$$
\lim_{n\to\infty}\left(\frac{\sum_{i=1}^{n}X_i}{b_n} - n^{1-\alpha}\tilde{L}^{-1}(n^\alpha)\mathbb{E}X\right) = \lim_{n\to\infty}\frac{\sum_{i=1}^{n}(X_i-\mathbb{E}X_i)}{b_n} = 0 \quad \text{a.s.} \tag{2.21}
$$

Similarly, we obtain from (2.16) that

$$
\lim_{n\to\infty}\left(\frac{\sum_{i=1}^{n}X_i}{b_n} - n^{1-\alpha}\tilde{L}^{-1}(n^\alpha)c\right) = 0 \quad \text{a.s.} \tag{2.22}
$$

Combining (2.21) and (2.22) yields

$$
\lim_{n\to\infty} n^{1-\alpha}\tilde{L}^{-1}(n^\alpha)(\mathbb{E}X - c) = 0 \quad \text{a.s.} \tag{2.23}
$$

If $\alpha < 1$, then $n^{1-\alpha}\tilde{L}^{-1}(n^\alpha) \to \infty$. If $\alpha = 1$, then by (2.3), we have $n^{1-\alpha}\tilde{L}^{-1}(n^\alpha) = \tilde{L}^{-1}(n) \sim L(n\tilde{L}(n)) \geq 1$. Therefore, we conclude from (2.23) that $c = \mathbb{E}X$. □

3. Proof of Theorem 1.1. In this section, we will present a result on the stochastic domination condition via regularly varying functions theory, and use it to prove Theorem 1.1. We need the following simple lemma. See Rosalsky and Thành [21] for a proof.

LEMMA 3.1. *Let $g : [0,\infty) \to [0,\infty)$ be a measurable function with $g(0) = 0$ which is bounded on $[0, A]$ and differentiable on $[A,\infty)$ for some $A \geq 0$. If ξ is a nonnegative random variable, then*

$$
\mathbb{E}(g(\xi)) = \mathbb{E}(g(\xi)\mathbf{1}(\xi \leq A)) + g(A) + \int_A^\infty g'(x)\mathbb{P}(\xi > x)\mathrm{d}x. \tag{3.1}
$$

The following result generalizes and unifies Theorem 2.5 (ii) and (iii) of Rosalsky and Thành [21]. The proof is similar to that of Theorem 2.6 in [25].

THEOREM 3.2. *Let $\{X_i, i \in I\}$ be a family of random variables and let $L(\cdot)$ be a slowly varying function. If*

$$\sup_{i\in I} \mathbb{E}\left(|X_i|^p L(|X_i|)\log(|X_i|)\log^2(\log|X_i|)\right) < \infty \quad \text{for some } p > 0, \tag{3.2}$$

then there exists a nonnegative random variable X with distribution function $F(x) = 1 - \sup_{i\in I}\mathbb{P}(|X_i| > x)$, $x \in \mathbb{R}$ such that $\{X_i, i \in I\}$ is stochastically dominated by X and

$$\mathbb{E}\left(X^p L(X)\right) < \infty. \tag{3.3}$$

Proof. By (3.2) and Theorem 2.5 (i) of Rosalsky and Thành [21], we get that $\{X_i, i \in I\}$ is stochastically dominated by a nonnegative random variable X with distribution function

$$F(x) = 1 - \sup_{i\in I}\mathbb{P}(|X_i| > x),\ x \in \mathbb{R}.$$

Let

$$g(x) = x^p L(x)\log(x)\log^2(\log x),\ h(x) = x^p L(x),\ x \ge 0.$$

Applying the first half of (2.4), there exists B large enough such that $g(\cdot)$ and $h(\cdot)$ are strictly increasing on $[B, \infty)$, and

$$\left|\frac{xL'(x)}{L(x)}\right| \le \frac{p}{2},\ x > B.$$

Therefore,

$$h'(x) = px^{p-1}L(x) + x^p L'(x) = x^{p-1}L(x)\left(p + \frac{xL'(x)}{L(x)}\right) \le \frac{3px^{p-1}L(x)}{2},\ x > B. \tag{3.4}$$

By Lemma 3.1, (3.2) and (3.4), there exists a constant C_1 such that

$$\begin{aligned}
\mathbb{E}(h(X)) &= \mathbb{E}(h(X)\mathbf{1}(X \le B)) + h(B) + \int_B^\infty h'(x)\mathbb{P}(X > x)\mathrm{d}x \\
&\le C_1 + \frac{3p}{2}\int_B^\infty x^{p-1}L(x)\mathbb{P}(X > x)\mathrm{d}x \\
&= C_1 + \frac{3p}{2}\int_B^\infty x^{p-1}L(x)\sup_{i\in I}\mathbb{P}(|X_i| > x)\mathrm{d}x \\
&\le C_1 + \frac{3p}{2}\int_B^\infty x^{-1}\log^{-1}(x)\log^{-2}(\log x)\sup_{i\in I}\mathbb{E}\left(g(|X_i|)\right)\mathrm{d}x \\
&= C_1 + \frac{3p}{2}\sup_{i\in I}\mathbb{E}\left(g(|X_i|)\right)\int_B^\infty x^{-1}\log^{-1}(x)\log^{-2}(\log x)\mathrm{d}x \\
&< \infty.
\end{aligned}$$

The proposition is proved. □

Remark 4. The contribution of the slowly varying function $L(x)$ in Theorem 3.2 helps us to unify Theorem 2.5 (ii) and (iii) of Rosalsky and Thành [21]. Letting $L(x) = \log^{-1}(x)\log^{-2}(\log x)$, $x \geq 0$, then by Theorem 3.2, the condition

$$\sup_{i\in I} \mathbb{E}|X_i|^p < \infty \quad \text{for some } p > 0,$$

implies that the family $\{X_i, i \in I\}$ is stochastically dominated by a nonnegative random variable X satisfying

$$\mathbb{E}\left(X^p \log^{-1}(X)\log^{-2}(\log X)\right) < \infty.$$

This slightly improves Theorem 2.5 (ii) in Rosalsky and Thành [21]. Similarly, by letting $L(x) = 1$, we obtain an improvement of Theorem 2.5 (iii) in Rosalsky and Thành [21].

We now recall a two-sided inequality stated in [4] to derive the necessary conditions for the validity of the weak law of large numbers under appropriate dependence restrictions. In the following lemma, we apply Theorem 2.3 in [4] for random variables $X_n' = X_n - c_n, n \geq 1$.

Lemma 3.3. *Let $\{X_n,\ n \geq 1\}$ be a sequence of random variables fulfilling* (1.2) *and let $\{c_n, n \geq 1\}$ be a sequence of real numbers. For $t > 0$ and $n \geq 1$, put*

$$I_n(t) = \mathbb{P}(\max_{1\leq i\leq n} |X_i - c_i| > t) \quad \text{and} \quad J_n(t) = \sum_{i=1}^{n} \mathbb{P}(|X_i - c_i| > t).$$

Then

$$\frac{1}{2}\cdot\frac{J_n(t)}{2C + J_n(t)} \leq I_n(t) \leq J_n(t),\ n \geq 1,$$

where C is given by (1.2). *In particular, if $I_n(u_n) = o(1)$ for some positive sequence $\{u_n,\ n \geq 1\}$, then $I_n(u_n) \asymp J_n(u_n)$.*

Proof of Theorem 1.1. By applying Theorem 3.2, we have from (1.3) that the sequence $\{X_n, n \geq 1\}$ is stochastically dominated by a nonnegative random variable X with

$$\mathbb{E}\left(|X|^p L^p(|X|)\right) < \infty.$$

Applying Theorem 2.1, we immediately obtain (1.4).

We now turn to the proof of the second part of the theorem. Assume that (1.5) is met. Let $b_n = n^\alpha \tilde{L}\left(n^\alpha\right), n \geq 1$, and let $\varepsilon > 0$ be arbitrary. For $n \geq 1$ and $t > 0$, let $S_0 = 0$, $S_n = \sum_{i=1}^n (X_i - c_i)$, $I_n(t)$ and $J_n(t)$ be as in Lemma 3.3. From the relation $|X_n - c_n| \leq |S_n| + |S_{n-1}|$, $n \geq 1$, we infer that

$$I_n(\varepsilon b_n) \leq \mathbb{P}\left(\max_{1\leq k\leq n} |S_k| > \varepsilon\frac{b_n}{2}\right).$$

Joining this with (1.5), we reach that

$$\sum_{n\geq 1} n^{\alpha p-2} I_n(\varepsilon b_n) < \infty. \tag{3.5}$$

Besides, since the sequence $\{b_n, n \geq 1\}$ is increasing, we obtain from (3.5) that

$$n^{\alpha p-1} I_n(\varepsilon b_{2n}) \leq \sum_{k=n+1}^{2n} k^{\alpha p-2} I_k(\varepsilon b_k) = o(1),$$

which, in view of the range of α and Lemma 2.1(ii) of [6], leads to $I_n(\varepsilon b_n) = o(1)$. By invoking Lemma 3.3, we conclude that $I_n(\varepsilon b_n) \asymp J_n(\varepsilon b_n)$ implying, via (3.5),

$$\sum_{n\geq 1} n^{\alpha p-2} J_n(\varepsilon b_n) < \infty.$$

This establishes the thesis and achieves the proof of the theorem. □

The following example illustrates the sharpness of Theorem 1.1 (and Corollary 1.2). It shows that in Theorem 1.1, (1.4) may fail if (1.3) is weakened to

$$\sup_{n\geq 1} \mathbb{E}\left(|X_n|^p L^p(|X_n|) \log(|X_n|) \log(\log |X_n|)\right) < \infty. \tag{3.6}$$

It therefore also shows that the main result of Sung [23, Theorem 3.1] may fail if the underlying random variables are not identically distributed.

Example 3.1. Let $1 \leq p < 2$ and $L(\cdot)$ be a positive slowly varying function such that $g(x) = x^p L^p(x)$ is strictly increasing on $[A, \infty)$ for some $A > 0$. Let $B = \lfloor A+g(A) \rfloor + 1$, $h(x)$ be the inverse function of $g(x)$, $x \geq B$, and let $\{X_n, n \geq B\}$ be a sequence of independent random variables such that for all $n \geq B$

$$\mathbb{P}(X_n = 0) = 1 - \frac{1}{n\log(n)\log(\log n)},\ \mathbb{P}\left(X_n = \pm h(n)\right) = \frac{1}{2n\log(n)\log(\log n)}.$$

By (2.3), we can choose (unique up to asymptotic equivalence)

$$\tilde{L}(x) = \frac{h(x^p)}{x},\ x \geq B.$$

Since $\tilde{L}(\cdot)$ is a slowly varying function,

$$\log(\tilde{L}(n^{1/p})) = o\left(\log n\right),$$

and so

$$\log h(n) = \log\left(n^{1/p}\tilde{L}(n^{1/p})\right) = \frac{1}{p}\log n + o(\log n).$$

It thus follows that

$$\begin{aligned}
&\sup_{n\geq 1} \mathbb{E}\left(|X_n|^p L^p(|X_n|) \log(|X_n|) \log^2(\log |X_n|)\right) \\
&= \sup_{n\geq 1} \mathbb{E}\left(g(|X_n|) \log(|X_n|) \log^2(\log |X_n|)\right) \\
&= \sup_{n\geq 1} \frac{\log(h(n)) \log^2(\log h(n))}{\log(n)\log(\log n)} = \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{n\geq 1}\mathbb{E}\left(|X_n|^p L^p(|X_n|)\log(|X_n|)\log(\log|X_n|)\right)\\
&=\sup_{n\geq 1}\mathbb{E}\left(g(|X_n|)\log(|X_n|)\log(\log|X_n|)\right)\\
&=\sup_{n\geq 1}\frac{\log(h(n))\log(\log h(n))}{\log(n)\log(\log n)}<\infty.
\end{aligned}$$

Therefore (1.3) fails but (3.6) holds.

Now, if (1.4) holds, then by letting $\alpha = 1/p$, we have

$$\lim_{n\to\infty}\frac{\sum_{i=B}^{n}X_i}{n^{1/p}\tilde{L}(n^{1/p})}=0 \text{ a.s.} \tag{3.7}$$

It follows from (3.7) that

$$\lim_{n\to\infty}\frac{X_n}{n^{1/p}\tilde{L}(n^{1/p})}=0 \text{ a.s.} \tag{3.8}$$

Since the sequence $\{X_n, n\geq 1\}$ is comprised of independent random variables, the Borel–Cantelli lemma and (3.8) ensure that

$$\sum_{n=B}^{\infty}\mathbb{P}\left(|X_n|>n^{1/p}\tilde{L}(n^{1/p})/2\right)<\infty. \tag{3.9}$$

However, we have

$$\begin{aligned}
\sum_{n=B}^{\infty}\mathbb{P}\left(|X_n|>n^{1/p}\tilde{L}(n^{1/p})/2\right)&=\sum_{n=B}^{\infty}\mathbb{P}\left(|X_n|>h(n)/2\right)\\
&=\sum_{n=B}^{\infty}\frac{1}{n\log(n)\log(\log n)}=\infty
\end{aligned}$$

contradicting (3.9). Therefore, (1.4) must fail.

Now, if we choose $L(x)\equiv 1$, then all assumptions of Theorem 3.1 of Sung [23] are met except for the identical distribution hypothesis. It follows from the above argument that (3.7) also fail (with $\tilde{L}(x)\equiv 1$). Therefore, this shows that Theorem 3.1 of Sung [23] may fail if the underlying random variables are not identically distributed.

4. Corollaries and remarks. In this section, we apply Theorems 1.1, 2.1, 2.2 to sequences of (i) m-pairwise negatively dependent random variables and (ii) extended negatively dependent random variables. The results obtained in this section are new even when $L(x)\equiv 1$. We also present some remarks to compare our results with the existing ones.

4.1. m-pairwise negatively dependent random variables. The Baum–Katz theorem and the Marcinkiewicz–Zygmund SLLN for sequences of m-pairwise negatively dependent random variables were studied by Wu and Rosalsky [27]. Let $m \geq 1$ be a fixed integer. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be *m-pairwise negatively dependent* if for all positive integers j and k with $|j-k| \geq m$, X_j and X_k are negatively dependent, i.e.,

$$\mathbb{P}(X_j \leq x, X_k \leq y) \leq \mathbb{P}(X_j \leq x)\mathbb{P}(X_k \leq y) \text{ for all } x, y \in \mathbb{R}.$$

When $m = 1$, this reduces to the usual concept of pairwise negative dependence. It is well known that if $\{X_n, n \geq 1\}$ is a sequence of m-pairwise negatively dependent random variables and $\{f_n, n \geq 1\}$ is a sequence of nondecreasing functions, then $\{f_n(X_n), n \geq 1\}$ is a sequence of m-pairwise negatively dependent random variables.

The following corollary is the first result in the literature on the complete convergence for sequences of m-pairwise negatively dependent random variables under the optimal condition even when $m = 1$ and $L(x) \equiv 1$.

Corollary 4.1. *Let $1 \leq p < 2$, $\alpha \geq 1/p$, and let $\{X_n,\, n \geq 1\}$ be a sequence of m-pairwise negatively dependent random variables, and $L(\cdot)$ as in Theorem 1.1.*

(i) *If* (1.3) *holds, then we obtain* (1.4).

(ii) *If $\{X_n,\, n \geq 1\}$ is stochastically dominated by a random variable X satisfying* (2.5), *then we obtain* (2.6). *Conversely, if $\alpha \leq 1$ and the random variables $X, X_1, X_2, \ldots$ are identically distributed, then* (2.6) *implies* (2.5).

Proof. From Lemma 2.1 in Wu and Rosalsky [27], it is easy to see that m-pairwise negatively dependent random variables satisfy Condition (1.2). Therefore, Part (i) follows from Theorems 1.1, and Part (ii) follows from Theorems 2.1 and 2.2. □

Remark 5. (i) We consider a special case where $\alpha = 1/p$, $1 < p < 2$ and $L(x) \equiv 1$ in Corollary 4.1 (ii). Under the condition $\mathbb{E}(|X|^p) < \infty$, we obtain

$$\sum_{n=1}^{\infty} n^{-1}\mathbb{P}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(X_i - \mathbb{E}X_i)\right| > \varepsilon n^{1/p}\right) < \infty \text{ for all } \varepsilon > 0, \tag{4.1}$$

and therefore

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n}(X_i - \mathbb{E}X_i)}{n^{1/p}} = 0 \text{ a.s.} \tag{4.2}$$

(ii) For $1 < p < 2$, Sung [23] considered the pairwise independent case and obtained (4.2) under a slightly stronger condition that

$$\mathbb{E}\left(|X|^p(\log\log|X|)^{2(p-1)}\right) < \infty. \tag{4.3}$$

Furthermore, one cannot obtain the rate of convergence (4.1) by using the method used in Sung [23]. In Chen et al. [7, Theorem 3.6], the authors proved (4.1) assuming $\mathbb{E}(|X|^p \log^r |X|) < \infty$ for some $r > p$. They raised an open question as to whether or not (4.1) holds under (4.3) (see [7, Remark 3.1]). For the case where the random variables are m-pairwise negatively dependent, Wu and Rosalsky [27] obtained (4.1) and (4.2) under the condition $\mathbb{E}(|X|^p \log^r |X|) < \infty$ for some $r > 1 + p$. Wu and Rosalsky [27] then raised an open question as to whether or not (4.2) holds under Condition (4.3). For $p = 1$ and the underlying random variables are m-pairwise negatively dependent, Wu and Rosalsky [27, Remarks 3.6] stated another open question as to whether or not (4.1) (with $p = 1$) holds under the condition $\mathbb{E}|X| < \infty$. Therefore, a very special case of Corollary 4.1 gives affirmative answers to the mentioned open questions raised by Chen et al. [7] and Wu and Rosalsky [27].

4.2. Extended negatively dependent random variables. The Kolmogorov SLLN for extended negatively dependent was first studied by Chen et al. [8]. A collection of random variables $\{X_1, \ldots, X_n\}$ is said to be *extended negatively dependent* if for all $x_1, \ldots, x_n \in \mathbb{R}$, there exists $M > 0$ such that

$$\mathbb{P}(X_1 \le x_1, \ldots, X_n \le x_n) \le M\mathbb{P}(X_1 \le x_1) \ldots \mathbb{P}(X_n \le x_n),$$

and

$$\mathbb{P}(X_1 > x_1, \ldots, X_n > x_n) \le M\mathbb{P}(X_1 > x_1) \ldots \mathbb{P}(X_n > x_n).$$

A sequence of random variables $\{X_i, i \ge 1\}$ is said to be extended negatively dependent if for all $n \ge 1$, the collection $\{X_i, 1 \le i \le n\}$ is extended negatively dependent.

Let m be a positive integer. The notion of m-extended negative dependence was introduced in Wu and Wang [28]. A sequence $\{X_i, i \ge 1\}$ of random variables is said to be m-extended negatively dependent if for any $n \ge 2$ and any $i_1, i_2, \ldots, i_n$ such that $|i_j - i_k| \ge m$ for all $1 \le j \le k \le n$, we have $\{X_{i_1}, \ldots, X_{i_n}\}$ are extended negatively dependent. If $\{X_i, i \ge 1\}$ is a sequence of m-extended negatively dependent random variables and $\{f_i, i \ge 1\}$ is a sequence of nondecreasing functions, then $\{f_i(X_i), i \ge 1\}$ is a sequence of m-extended negatively dependent random variables. We note that the classical Kolmogorov maximal inequality or the classical Rosenthal maximal inequality are not available for extended negatively dependent random variables (see Wu and Wang [28], Wu et al. [29]).

COROLLARY 4.2. *Corollary 4.1 holds if $\{X_n, n \ge 1\}$ is a sequence of m-extended negatively dependent random variables.*

Proof. Lemma 3.3 of Wu and Wang [28] implies that the sequence $\{X_n, n \ge 1\}$ satisfies Condition (1.2). Corollary 4.2 then follows from Theorems 1.1, 2.1, and 2.2. □

REMARK 6. Chen et al. [8] proved the Kolmogorov SLLN for sequences of extended negatively dependent and identically distributed random variables $\{X, X_n, n \ge 1\}$ under the condition that $\mathbb{E}|X| < \infty$. They used Etemadi's method in Etemadi [11] which does not seem to work for the case of the Marcinkiewicz–Zygmund SLLN.

To our best knowledge, Corollary 4.2 is the first result in the literature on the Baum–Katz theorem for sequences of m-extended negatively dependent random variables under the optimal moment condition even when $L(x) \equiv 1$ and $m = 1$.

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